On a relation between the Bach equation and the equation of geometrodynamics

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Abstract

The Bach equation and the equation of geometrodynamics are based on two quite different physical motivations, but in both approaches, the conformal properties of gravitation plays the key role. In this paper we present an analysis of the relation between these two equations and show that the solutions of the equation of geometrodynamics are of a more general nature. We show the following nontrivial result: there exists a conformally invariant Lagrangian, whose field equation generalizes the Bach equation and has as solutions those Ricci tensors which are solutions to the equation of geometrodynamics.

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1 Introduction

Conformal properties for theories of gravity appear on several different levels. The best known example is the conformally coupled scalar field, where the conformal property is due to the fact, that in four dimensions, the operator $\Box - R/6$ is conformally invariant.

Two other examples are: First, the Bach equation $B_{\alpha\beta} = 0$, (see e.g. [1] and [2] and the references there for details, [3], [4] for further recent approaches to the Bach equation, and [5] for an overview on fourth-order gravity in the time period 1918 - 1990) whose conformal invariance follows from the conformal invariance of the Weyl tensor $C^{\alpha}{}_{\beta\gamma\delta}$, and, second, the equation of geometrodynamics, proposed in 1984 in [6], see also [7] and [8]. As the details of the latter are not so well-known, we sum up their properties in the Appendix.

It is the purpose of the present paper to compare these two latter approaches.

2 Notation

We apply the following notation: Riemann tensor:

$$R^{\lambda}{}_{\sigma\alpha\beta} = \Gamma^{\lambda}_{\beta\sigma,\alpha} - \ldots \tag{1}$$

Ricci tensor:

$$R_{\alpha\beta} = R^{\sigma}{}_{\ \alpha\sigma\beta} \,. \tag{2}$$

This is the rule for rearrangement of order of covariant differentiation:

$$Y^{\lambda}{}_{;\alpha;\beta} = Y^{\lambda}{}_{;\beta;\alpha} - Y^{\sigma} R^{\lambda}{}_{\sigma\alpha\beta}; \quad Y_{\lambda;\alpha;\beta} = Y_{\lambda;\beta;\alpha} + Y_{\sigma} R^{\sigma}{}_{\lambda\alpha\beta} \,. \tag{3}$$

The Weyl tensor is:

$$\begin{aligned} C_{\alpha\beta\mu\nu} \equiv R_{\alpha\beta\mu\nu} - \tfrac{1}{2}\left[g_{\alpha\mu}R_{\beta\nu} + g_{\beta\nu}R_{\alpha\mu} - g_{\alpha\nu}R_{\beta\mu} - g_{\beta\mu}R_{\alpha\nu}\right] + \\ + \tfrac{1}{6}R\left[g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right] \,. \end{aligned} \tag{4}$$

3 The Bach equation

Bach equation describes the vanishing of the variational derivative of the square of the Weyl tensor with respect to the metric. We use it here as given in [2], i.e.:

$$B_{\alpha\beta} \equiv B^{(1)}_{\alpha\beta} + B^{(2)}_{\alpha\beta} = 0\,. \tag{5}$$

Here:

$$B^{(1)}_{\alpha\beta}\,(R_{..};g_{..}) = -R_{\alpha\beta}{}^{;\nu}{}_{;\nu} + R^{\nu}_{\alpha;\beta;\nu} + R^{\nu}_{\beta;\alpha;\nu} - \frac{2}{3}R_{;\alpha;\beta} + \frac{1}{6}g_{\alpha\beta}R^{;\nu}{}_{;\nu}\,, \tag{6}$$

$$B^{(2)}_{\alpha\beta}\,(R_{..};g_{..}) = \frac{2}{3}RR_{\alpha\beta} - 2R_{\alpha\nu}R^{\nu}{}_{\beta} - \frac{1}{6}R^2 g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}R_{\mu\nu}R^{\mu\nu}\,. \tag{7}$$

From eqs. (6), (7) it becomes clear that the quantities $B^{(1)}_{\alpha\beta}$, $B^{(2)}_{\alpha\beta}$ depend on the Ricci tensor and on the metric. It should be mentioned, that up to a multiplication of the whole Bach tensor by a non-vanishing constant, all these 9 coefficients in front of the 9 different geometric terms in the right hand sides of eqs. (6) and (7) are uniquely determined by the conformal and covariant properties of the Weyl tensor.

4 The equation of geometrodynamics

By the "equation of geometrodynamics" we denote the equation deduced in [6]; it reads (cf. the Appendix for a deduction):

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -2A_\alpha A_\beta - g_{\alpha\beta}A^2 - 2g_{\alpha\beta}A^{\nu}{}_{;\nu} + A_{\alpha;\beta} + A_{\beta;\alpha}\,, \tag{8}$$

where

$$A^2 = A_\mu A_\nu g^{\mu\nu}\,.$$

From the equation (8) we get after taking the divergence

$$F_\alpha{}^{\nu}{}_{;\nu} = 0\,, \tag{9}$$

where

$$F_{\alpha\beta} \equiv A_{\beta,\alpha} - A_{\alpha,\beta} = A_{\beta;\alpha} - A_{\alpha;\beta}\,. \tag{10}$$

The trace of equation (8) yields:

$$R = 6A^2 + 6A^{\nu}{}_{;\nu}\,. \tag{11}$$

Then the equation (8) can also be written as

$$R_{\alpha\beta} = -2A_\alpha A_\beta + 2g_{\alpha\beta}A^2 + g_{\alpha\beta}A^{\nu}{}_{;\nu} + A_{\alpha;\beta} + A_{\beta;\alpha}\,. \tag{12}$$

5 The definition of the problem

We apply the following conformal transformation:

$$g_{\alpha\beta}\left(x\right) \to g'{}_{\alpha\beta}\left(x\right) = g_{\alpha\beta}\left(x\right) \cdot e^{2\sigma(x)}\,. \tag{13}$$

The equation of geometrodynamics (8) is invariant with respect to conformal transformations (13), if the vector $A_\alpha\left(x\right)$ is simultaneously transformed as:

$$A_\alpha\left(x\right) \to A'{}_\alpha\left(x\right) = A_\alpha\left(x\right) - \sigma_{,\alpha}\left(x\right)\,. \tag{14}$$

The Bach equation (5) has the following solutions:

– Riemannian spaces with zero Ricci tensor, i.e. with $R_{\alpha\beta} = 0$. After a conformal transformation we get

$$R_{\alpha\beta} = -2\varphi_{,\alpha}\varphi_{,\beta} + 2g_{\alpha\beta}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} + g_{\alpha\beta}\varphi^{;\nu}{}_{;\nu} + \varphi_{;\alpha;\beta} + \varphi_{;\beta;\alpha}\,. \tag{15}$$

Here $\varphi\left(x\right)$ is an arbitrary function of the coordinates. Due to the conformal invariance of the Bach equation, Riemannian spaces with Ricci tensor (15) also satisfy the Bach equation. [1]

– Einstein spaces,

$$R_{\alpha\beta} = \lambda \cdot g_{\alpha\beta}, \quad \lambda \neq 0, \quad \lambda = \text{ const.}\,, \tag{16}$$

[1] Hint for a proof: Insert $\varphi = \sigma$ into eq. (13), then eq. (15) implies $R'_{\alpha\beta} \equiv 0$. The last two terms of eq. (15) coincide, of course, but we wrote them this way we did to ease the comparison with eq. (12).

as well as Riemannian spaces, that differ from the Einstein spaces by the conformal transformation (13). If this σ is a constant function $\sigma = \sigma_0 =$ const., then we get again an Einstein space, this time with $\lambda' = \lambda e^{-2\sigma_0}$.[2]

As a result, we have two types of conformally invariant equations that describe Riemannian space dynamics. Each type of equations can be treated as a basis for a conformally invariant theory of gravitation. These theories are not equivalent, since there are the following differences between Bach equations and equations of geometrodynamics:

1. The equation (5) is of the 4^{th} order, while (8) is a second-order equation.

2. The equation (8) includes the vector $A_\alpha(x)$, and the eq. (5) does not.

3. For the equation (8) a correctly-defined Cauchy problem exists (see [7]). As for the equation (5), correctness of the Cauchy problem definition has not been proved yet.[3]

4. The equation (5) can be derived from the variation principle; no Lagrangian with the dimension of $[length^2]$ exists for the equation (8).

Remember that the Lagrangian for the Bach equation reads:

$$L = C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} \,. \tag{17}$$

Taking into account the fact that

$$L_{GB} = R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\alpha\beta} R^{\alpha\beta} + R^2 \tag{18}$$

[2]Solutions of the Bach equation which are not conformally related to an Einstein space are called "non-trivial solutions", there existence is shown in [1], [3] and [4].

[3]More concretely, there are at least two open questions in this respect: First, that point in the usual Cauchy problem formulation of the Bach equation, where one breaks the conformal invariance by requiring the curvature scalar to be a constant: it is not clear whether this restricts globally the set of solutions or not; second, even for the simplest anisotropic cosmological models, namely Bianchi type I, it is not clear whether the statement "Without loss of generality, one may assume the Bianchi type I metric in diagonal form to get the full set of Bianchi type I vacuum solutions." proven for Einstein's theory, is valid for the Bach equation, too. This second question shall be discussed more detailed in a further paper.

is the complete Gauss-Bonnet divergence, the Lagrangian for the equation (5) can also be written as [4]:

$$\tilde{L} = 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2 \,. \tag{19}$$

It is clear that any solution of the Bach equation in the form (15) with a constant $\lambda = 0$ [5] will also satisfy the equation (8), since (15) can be reduced to (8) by equating

$$A_\alpha(x) = -\varphi_{,\alpha}(x) \quad . \tag{20}$$

The subject of this paper is the solution of the inverse problem, i.e. to find out under what conditions solutions of the equation of geometrodynamics (8) are also solutions to the Bach equation (5).

6 Solution of the problem

The Ricci tensor (12) will be referred to as geometrodynamical tensor and denoted as $R^{\mathrm{GD}}_{..}$. We will show that for $R^{\mathrm{GD}}_{..}$ there exists the following relation:

$$B_{\alpha\beta}\left(R^{\mathrm{GD}}_{..}; g_{..}\right) = 2F_{\alpha\nu}F_\beta{}^\nu - \frac{1}{2}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}. \tag{21}$$

The proof is lengthy. To make the procedure convenient for check, we will divide the proof into 7 steps.

Step 1.

In this step $R^{\mathrm{GD}}_{..}$ from (12) is substituted to $B^{(1)}_{\alpha\beta}(R_{..}; g_{..})$. This yields:

$$B^{(1)}_{\alpha\beta}\left(R^{\mathrm{GD}}_{..}\cdot\cdot\right) = \Phi_{\alpha\beta}(A''') + \Phi_{\alpha\beta}(AA'') + \Phi_{\alpha\beta}(A'A') \,. \tag{22}$$

The terms' dependence on the vector A_α and its derivatives $A_{\alpha;\beta}$ is shown

[4] up to surface terms in the action which do not influence the classical field equation

[5] The equation (8) can also be written with a non-zero λ-term, and with the transformations (13), (14), λ will be transformed as $\lambda \to \lambda' = \lambda \cdot e^{-2\sigma}$. For our proof it is, however, sufficient to assume $\lambda = 0$.

symbolically in brackets. In the explicit form, these terms read:

$$\begin{array}{l}\Phi_{\alpha\beta}\left(A'''\right)=-A^{\nu}{}_{;\nu;\alpha;\beta}-A^{\nu}{}_{;\nu;\beta;\alpha}+\\+A_{\alpha}{}^{;\nu}{}_{;\beta;\nu}+A_{\beta}{}^{;\nu}{}_{;\alpha;\nu}+A^{\nu}{}_{;\alpha;\beta;\nu}+A^{\nu}{}_{;\beta;\alpha;\nu}-A_{\alpha;\beta}{}^{;\nu}{}_{;\nu}-A_{\beta;\alpha}{}^{;\nu}{}_{;\nu}\end{array}. \qquad (23)$$

$$\begin{array}{l}\Phi_{\alpha\beta}\left(AA''\right)=2\left(A_{\alpha}{}^{;\nu}{}_{;\nu}\right)A_{\beta}+2\left(A_{\beta}{}^{;\nu}{}_{;\nu}\right)A_{\alpha}-2g_{\alpha\beta}A^{\sigma}\left(A_{\sigma}{}^{;\nu}{}_{;\nu}\right)-\\-2A_{\alpha}A^{\nu}_{;\beta;\nu}-2A_{\beta}A^{\nu}_{;\alpha;\nu}-2A^{\nu}A_{\alpha;\beta;\nu}-2A^{\nu}A_{\beta;\alpha;\nu}\end{array} \qquad (24)$$

$$\begin{array}{l}\Phi_{\alpha\beta}\left(A'A'\right)=4A_{\alpha;\nu}A_{\beta}{}^{;\nu}-2A_{\alpha;\nu}A^{\nu}{}_{;\beta}-2A_{\beta;\nu}A^{\nu}{}_{;\alpha}-\\-2A_{\alpha;\beta}\left(A^{\nu}{}_{;\nu}\right)-2A_{\beta;\alpha}\left(A^{\nu}{}_{;\nu}\right)-2g_{\alpha\beta}\left(A^{\mu;\nu}A_{\mu;\nu}\right)\end{array} \qquad (25)$$

Step 2.

In this step, the term $\Phi_{\alpha\beta}\left(A'''\right)$ is first transformed with the Bianchi identity to the following form:

$$\Phi_{\alpha\beta}\left(A'''\right)=2A^{\sigma}R^{\mathrm{GD}}_{\alpha\beta;\sigma}+2A^{\nu}{}_{;\alpha}R^{\mathrm{GD}}_{\nu\beta}+2A^{\nu}{}_{;\beta}R^{\mathrm{GD}}_{\nu\alpha}, \qquad (26)$$

then, after the expression for $R^{\mathrm{GD}}_{..}$ has been substituted, it reads:

$$\Phi_{\alpha\beta}\left(A'''\right)=\Theta_{\alpha\beta}\left(AA''\right)+\Theta_{\alpha\beta}\left(A'A'\right)+\Theta_{\alpha\beta}\left(A^{2}A'\right), \qquad (27)$$

where

$$\Theta_{\alpha\beta}\left(AA''\right)=2g_{\alpha\beta}A^{\nu}A^{\sigma}{}_{;\sigma;\nu}+2A^{\nu}A_{\alpha;\beta;\nu}+2A^{\nu}A_{\beta;\alpha;\nu}, \qquad (28)$$

$$\begin{array}{l}\Theta_{\alpha\beta}\left(A'A'\right)=2A_{\alpha;\beta}\left(A^{\nu}{}_{;\nu}\right)+2A_{\beta;\alpha}\left(A^{\nu}{}_{;\nu}\right)+4A_{\nu;\alpha}A^{\nu}{}_{;\beta}+\\+2A_{\nu;\alpha}A_{\beta}{}^{;\nu}+2A_{\nu;\beta}A_{\alpha}{}^{;\nu}\end{array}, \qquad (29)$$

$$\begin{array}{l}\Theta_{\alpha\beta}\left(A^{2}A'\right)=-4A^{\nu}A_{\alpha;\nu}A_{\beta}-4A^{\nu}A_{\beta;\nu}A_{\alpha}+8g_{\alpha\beta}\left(A^{\mu}A^{\nu}A_{\mu;\nu}\right)-\\-4A^{\nu}A_{\nu;\alpha}A_{\beta}-4A^{\nu}A_{\nu;\beta}+4A_{\alpha;\beta}A^{2}+4A_{\beta;\alpha}A^{2}\end{array}. \qquad (30)$$

Step 3.

In this step, all terms are substituted to the relation (22) for

$$B^{(1)}_{\alpha\beta}\left(R^{\mathrm{GD}}_{..};g_{..}\right);$$

here, $\Phi_{\alpha\beta}\left(A'''\right)$ being written in the form (27), where the relations for

$$\Theta_{\alpha\beta}\left(AA''\right);\quad\Theta_{\alpha\beta}\left(A'A'\right);$$

and $\Theta_{\alpha\beta}\left(A^2A'\right)$ are given by (28), (29), (30), respectively. This yields:

$$B^{(1)}_{\alpha\beta}\left(R^{\rm GD}_{..};g_{..}\right)=\Psi_{\alpha\beta}\left(AA''\right)+\Psi_{\alpha\beta}\left(A^2A'\right)+\Psi_{\alpha\beta}\left(A'A'\right), \qquad (31)$$

where

$$\begin{array}{l}\Psi_{\alpha\beta}\left(AA''\right)=2\left(A_{\alpha}{}^{;\nu}{}_{;\nu}\right)A_{\beta}+2\left(A_{\beta}{}^{;\nu}{}_{;\nu}\right)A_{\alpha}-2g_{\alpha\beta}A^{\sigma}\left(A_{\sigma}{}^{;\nu}{}_{;\nu}\right)-\\ -2A_{\alpha}A^{\nu}{}_{;\beta;\nu}-2A_{\beta}A^{\nu}{}_{;\alpha;\nu}+2g_{\alpha\beta}A^{\sigma}A^{\nu}{}_{;\nu;\sigma}\end{array}, \qquad (32)$$

$$\begin{array}{l}\Psi_{\alpha\beta}\left(A^2A'\right)=4A_{\alpha;\beta}A^2+4A_{\beta;\alpha}A^2+8g_{\alpha\beta}\left(A^{\mu}A^{\nu}A_{\mu;\nu}\right)-\\ -4A^{\nu}A_{\alpha;\nu}A_{\beta}-4A^{\nu}A_{\beta;\nu}A_{\alpha}-4A^{\nu}A_{\nu;\alpha}A_{\beta}-4A^{\nu}A_{\nu;\beta}A_{\alpha}\end{array}, \qquad (33)$$

$$\Psi_{\alpha\beta}\left(A'A'\right)=4A_{\alpha;\nu}A_{\beta}{}^{;\nu}+2A_{\nu;\alpha}A^{\nu}{}_{;\beta}-2g_{\alpha\beta}\left(A^{\mu;\nu}A_{\mu;\nu}\right)\quad . \qquad (34)$$

In this step, the terms with the third-order derivatives disappear from the relation for $B^{(1)}_{\alpha\beta}\left(R^{\rm GD}_{..};g_{..}\right)$. The next step will be to exclude the second-order derivatives from $B^{(1)}_{\alpha\beta}\left(R^{\rm GD}_{..};g_{..}\right)$.

Step 4.

We transform the expression $\Psi_{\alpha\beta}\left(AA''\right)$, using the properties of the Riemann tensor. This yields:

$$\begin{array}{l}\Psi_{\alpha\beta}\left(AA''\right)=-2g_{\alpha\beta}A^2\left(A^{\nu}{}_{;\nu}\right)-4g_{\alpha\beta}\left(A^{\mu}A^{\nu}A_{\mu;\nu}\right)-\\ -2A_{\alpha}F_{\beta}{}^{\nu}{}_{;\nu}-2A_{\beta}F_{\alpha}{}^{\nu}{}_{;\nu}+2g_{\alpha\beta}A^{\sigma}F_{\sigma}{}^{\nu}{}_{;\nu}\end{array}. \qquad (35)$$

If the Ricci tensor is written as (12), then the vector A_α shall necessarily satisfy the equation (9). Taking this into account, (35) will change to

$$\Psi_{\alpha\beta}\left(AA''\right)=-2g_{\alpha\beta}A^2\left(A^{\nu}{}_{;\nu}\right)-4g_{\alpha\beta}\left(A^{\mu}A^{\nu}A_{\mu;\nu}\right)\quad . \qquad (36)$$

Note that at this step the terms with second-order derivatives have been replaced by the terms with first-order derivative.

Step 5.

We will find the resulting relation for $B^{(1)}_{\alpha\beta}\left(R^{\rm GD}_{..};g_{..}\right)$.

$$B^{(1)}_{\alpha\beta}\left(R^{\rm GD}_{..}{\cdot\cdot}\right)=\Omega_{\alpha\beta}\left(A^2A'\right)+\Omega_{\alpha\beta}\left(A'A'\right), \qquad (37)$$

where

$$\Omega_{\alpha\beta}\left(A^2A'\right) = 4A_{\alpha;\beta}A^2 + 4A_{\beta;\alpha}A^2 + 4g_{\alpha\beta}\left(A^\mu A^\nu A_{\mu;\nu}\right) - \\ -4A^\nu A_{\alpha;\nu}A_\beta - 4A^\nu A_{\beta;\nu}A_\alpha - 4A^\nu A_{\nu;\alpha}A_\beta - 4A^\nu A_{\nu;\beta}A_\alpha - 2g_{\alpha\beta}A^2\left(A^\nu{}_{;\nu}\right), \tag{38}$$

$$\Omega_{\alpha\beta}\left(A'A'\right) = 4A_{\alpha;\nu}A_\beta{}^{;\nu} + 4A_{\nu;\alpha}A^\nu{}_{;\beta} - 2g_{\alpha\beta}\left(A^{\mu;\nu}A_{\mu;\nu}\right) \quad . \tag{39}$$

Step 6.

We substitute the expression for $R^{\mathrm{GD}}_{..}$ in the relation for $B^{(2)}_{\alpha\beta}\left(R^{\mathrm{GD}}_{..};g_{..}\right)$. All algebraic terms (the terms that do not include derivative of the vector A_α) are eliminated, only terms with the derivatives of not higher than the first order remaining.

$$B^{(2)}_{\alpha\beta}\left(R^{\mathrm{GD}}_{..};g_{..}\right) = \Xi_{\alpha\beta}\left(A^2A'\right) + \Xi_{\alpha\beta}\left(A'A'\right), \tag{40}$$

here

$$\Xi_{\alpha\beta}\left(A^2A'\right) = -\Omega_{\alpha\beta}\left(A^2A'\right) \quad , \tag{41}$$

$$\Xi_{\alpha\beta}\left(A'A'\right) = -2A_{\alpha;\nu}A_\beta{}^{;\nu} - 2A_{\alpha;\nu}A^\nu{}_{;\beta} - 2A_{\beta;\nu}A^\nu{}_{;\alpha} \\ -2A_{\nu;\alpha}A^\nu{}_{;\beta} + g_{\alpha\beta}\left(A^{\mu;\nu}A_{\mu;\nu}\right) + g_{\alpha\beta}\left(A^{\mu;\nu}A_{\nu;\mu}\right) \quad . \tag{42}$$

Step 7.

The proof of the relation (21). To prove this relation, we must substitute the relation (37) for $B^{(1)}_{\alpha\beta}\left(R^{\mathrm{GD}}_{..};g_{..}\right)$ and the relation (40) for $B^{(2)}_{\alpha\beta}\left(R^{\mathrm{GD}}_{..};g_{..}\right)$ into the RHS of (21). So, the relation (21) has been proven. The relation (21) leads to the following three statements.

Statement 1:

If A_α is a gradient vector, i.e. it is in the form (20), then the Riemannian space with Ricci tensor (12) is a solution to the equation (5), since in this case $F_{\alpha\beta} = 0$.

Statement 2:

If we take a Lagrangian in the form[6] (43) instead of (19),

$$\tilde{\tilde{L}} = \tilde{L} + 2F_{\mu\nu}F^{\mu\nu} = 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2 + 2F^2, \tag{43}$$

[6] This Lagrangian will be referred to as Weyl-Maxwell Lagrangian (WM Lagrangian), see also [9] for it.

then the generalized Bach equation will read

$$B'_{\alpha\beta} \equiv B^{(1)}_{\alpha\beta} + B'^{(2)}_{\alpha\beta} = 0, \tag{44}$$

where $B^{(1)}_{\alpha\beta}$ is given by (6), and $B'^{(2)}_{\alpha\beta}$ is as follows:

$$\begin{array}{l} B'^{(2)}_{\alpha\beta} = B^{(2)}_{\alpha\beta} - 2F_{\alpha\nu}F^{\nu}{}_{\beta} - \frac{1}{2}g_{\alpha\beta}\left(F_{\mu\nu}F^{\mu\nu}\right) = \frac{2}{3}RR_{\alpha\beta} - \\ -2R_{\alpha\nu}R^{\nu}{}_{\beta} - \frac{1}{6}R^2 g_{\alpha\beta} + \frac{1}{2}g_{\alpha\beta}R_{\mu\nu}R^{\mu\nu} - 2F_{\alpha\nu}F^{\nu}{}_{\beta} - \frac{1}{2}g_{\alpha\beta}\left(F_{\mu\nu}F^{\mu\nu}\right) \end{array} \tag{45}$$

The Ricci tensor (12) satisfies the equation (44).

Statement 3:

If the Lagrangian (19) is replaced by the Lagrangian

$$\tilde{L} + \text{const} \cdot F_{\mu\nu}F^{\mu\nu} = 2R_{\alpha\beta}R^{\alpha\beta} - \frac{2}{3}R^2 + \text{const} \cdot F^2, \tag{46}$$

where const $\neq +2$, then the Ricci tensor (12) will only satisfy the dynamic equation if the vector A_α is in the gradient form.

7 The spherically symmetric solutions of the Bach equation and the geometrodynamics equation

Any spherically symmetric solution of the Bach equation is equivalent (almost everywhere and up to a conformal factor) to the Schwarzschild static solution in general relativity (GR). Each of these solutions in GR is governed by a single parameter, i.e. the gravitational radius.

For the geometrodynamics equation (8) a spherically symmetric solution can also be given in the static form. However, the spherically symmetric solution is now governed by two parameters, rather than one. We will write explicitly one of the possible static forms of the spherically symmetric solutions [7] of the equation (8). The peculiarity of this form is that the radial

[7] A detailed description of how to derive different forms of spherically symmetric solutions to geometrodynamic equation will be covered in a separate publication.

component is also the one that describes brightness.

$$
\begin{aligned}
ds^2 = &-\left(1+\tfrac{r}{R_0}\right)\left(1-\tfrac{r}{R_0}\right)\left(1-\tfrac{r_0}{r}\right)\cdot dt^2 + \frac{dr^2}{\left(1+\frac{r}{R_0}\right)\left(1-\frac{r}{R_0}\right)\left(1-\frac{r_0}{r}\right)} + \\
&+r^2\cdot\left[d\theta^2+\sin^2\theta\cdot d\varphi^2\right] .
\end{aligned}
\tag{47}
$$

At $r_0 < r << R_0$ the metric (47) is similar to the Schwarzschild metric, where r_0 works as a gravitational radius. It holds: For $R_0 = 0$, metric (47) is the exact Schwarzschild solution, for $r_0 = 0$, metric (47) is the exact de Sitter solution, but for $r_0 \cdot R_0 \neq 0$, it does *not* give the Schwarzschild-de Sitter solution.

Meanwhile, the topology of the solution (47) is qualitatively different from the topology of the Schwarzschild solution. The difference is, that the solution (47) has a horizon not only at the surface

$$r = r_0\,, \tag{48}$$

but also a singularity of the conformal curvature invariant at the surface

$$r = R_0\,. \tag{49}$$

8 Discussion

We must point out some facts given by the obtained results.

• This paper illustrated, with a static spherically symmetric solution as an example, the correspondence between the solution of the Bach equation (the solution with the vector A_α given by $A_\alpha = -\varphi_{,\alpha}$) and that of the geometrodynamics equation (i.e. the solution with non-gradient vector A_α). The fact is that a solution involving a non-gradient vector A_α ($A_\alpha \neq -\varphi_{,\alpha}$) cannot coincide, in principle, with the Schwarzschild solution in the entire range of variation of variables, though in some interval of the radial coordinate it may come close to this solution.

• Solutions of eq. (8) describe dissipative phenomena like heat conduction and viscosity (see [8]). One of the results of this paper is the proven fact that

solutions of this type are also solutions to the equation (44), that can be derived from the holonomic variation principle by a conventional procedure. This fact itself is non-trivial. Another amazing feature is that the structure of the corresponding holonomic Lagrangian is unambiguous.

• It is likely that the structure of the Weyl-Maxwell Lagrangian in the form (43) enables the construction of an analogue of a Hamiltonian formalism for solutions of the equation (8) and to quantize, despite the fact that, this equation describes dissipative phenomena.

• The addition of the Maxwellian term ($F_{\mu\nu}F^{\mu\nu}$) to the Lagrangian (43) does not immediately allow to interpret the vector A_α as an electromagnetic vector potential. The Lagrangian (43) does not yield the Einstein field equations with the Maxwell energy-momentum tensor, but the equation (44). The Lagrangian (43) includes two conformally invariant scalar values with Weyl weights, inversely proportional to $\sqrt{-g}$. These scalars are: $C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$ and $F_{\mu\nu}F^{\mu\nu}$. They give a complete list of the Weyl space invariants [8]. In fact, the Lagrangian that includes both invariants is the only possible one among those able to produce a dimensionless action with zero Weyl weight, in the case of Lagrangian being constructed immediately for the Weyl space. For this reason this represents a more natural interpretation for the vector A_α as being a Weyl vector.

Appendix: Deduction of the properties of geometrodynamics

Here we give a deduction of basic facts connected with the equation of geometrodynamics. This deduction is both a new approach and also more general than that ones given in the literature up to now.

Assume that the source for the Einstein field equation shall be composed

[8] The quantities $E_{\alpha\beta\mu\nu}C^{\alpha\beta\sigma\rho}C_{\sigma\rho}{}^{\mu\nu}$ and $E_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu}$ have the same Weyl weight, they are, however, pseudoscalars. Here, $E_{....}$ denotes the dual of the Weyl tensor.

from a vector field A_α, where at most first derivatives of it shall enter the energy-momentum tensor. A possible Λ-term can be omitted without loss of generality, [9] and terms of cubic and higher degree in A_α shall be omitted. Then we are left with the following equation

$$R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} = c_1\, A_\alpha A_\beta + c_2\, A_{(\alpha;\beta)} + c_3\, g_{\alpha\beta} A^2 + c_4\, g_{\alpha\beta} A^\gamma{}_{;\gamma} \tag{50}$$

containing the four free parameters c_i, $i = 1, \ldots, 4$. Here, $A^2 = A^\gamma A_\gamma$ and the round symmetrization bracket is defined via

$$A_{(\alpha;\beta)} = \frac{1}{2} \left(A_{\alpha;\beta} + A_{\beta;\alpha} \right) . \tag{51}$$

This equation (50) goes over to eq. (8), if the quadruple c_i takes the form

$$c_i = (-2,\, 2,\, -1,\, -2\,) . \tag{52}$$

The trace of eq. (50) reads

$$-\,R = (c_1 + 4c_3) A^2 + (c_2 + 4c_4) A^\gamma{}_{;\gamma} . \tag{53}$$

If one inserts eq. (52) into eq. (53), one gets, of course, eq. (11).

We take the covariant divergence of eq. (50). By use of the Bianchi identity, the l.h.s. identically vanishes, so we get

$$0 = c_1 A^\gamma{}_{;\gamma} A_\beta + c_1 A^\gamma A_{\beta;\gamma} + \frac{c_2}{2} A^\gamma{}_{;\beta\gamma} + \frac{c_2}{2} \Box A_\beta + 2c_3 A^\gamma A_{\gamma;\beta} + c_4 A^\gamma{}_{;\gamma\beta} . \tag{54}$$

For the further calculation it proves useful to define a second vector field, T_β, via the equation

$$T_\beta = A^\alpha\, R_{\alpha\beta} . \tag{55}$$

With eqs. (50) and (53) we find

$$T_\beta = c_2 A^\alpha A_{(\alpha;\beta)} + A_\beta \left((\frac{c_1}{2} - c_3) A^2 - (\frac{c_2}{2} + c_4) A^\gamma{}_{;\gamma} \right) . \tag{56}$$

To keep simplicity, we again require the absence of cubic terms, i.e.

$$c_1 = 2c_3 . \tag{57}$$

[9] With $\Lambda \neq 0$ the result would be analogous.

For the quadruple eq. (52) this condition is fulfilled.

Now we apply eq. (3) and find with eq. (55)

$$A^{\alpha}{}_{;\beta\alpha} - A^{\alpha}{}_{;\alpha\beta} = T_{\beta} \,. \tag{58}$$

Eq. (56) with eq. (57) gives

$$T_{\beta} = c_2 A^{\alpha} A_{(\alpha;\beta)} - (\frac{c_2}{2} + c_4) A_{\beta} A^{\gamma}{}_{;\gamma} \,. \tag{59}$$

Eq. (54) with eq. (57) gives

$$0 = 2c_3 A^{\gamma}_{;\gamma} A_{\beta} + 4c_3 A^{\gamma} A_{(\beta;\gamma)} + \frac{c_2}{2} A^{\gamma}_{;\beta\gamma} + \frac{c_2}{2} \Box A_{\beta} + c_4 A^{\gamma}_{;\gamma\beta} \,. \tag{60}$$

The goal is now to find out under which circumstances eq. (9) can be deduced from eqs. (58) till (60). Eq. (9) with eq. (10) reads

$$\Box A_{\beta} = A^{\gamma}_{;\beta\gamma} \,. \tag{61}$$

Therefore, the term $A^{\gamma}_{;\gamma\beta}$ of eq. (60) is superfluous. We take the three eqs. (58) till (60), cancel the vector T_{β} and the term $A^{\gamma}_{;\gamma\beta}$ and get afterwards:

$$\begin{aligned} 0 = 2c_3 A^{\gamma}_{;\gamma} A_{\beta} + 4c_3 A^{\gamma} A_{(\beta;\gamma)} + \frac{c_2}{2} A^{\gamma}{}_{;\beta\gamma} + \frac{c_2}{2} \Box A_{\beta} + \\ + c_4 A^{\gamma}_{;\beta\gamma} - c_2 c_4 A^{\gamma} A_{(\beta;\gamma)} + c_4 \left(\frac{c_2}{2} + c_4 \right) A_{\beta} A^{\gamma}{}_{;\gamma} \,. \end{aligned} \tag{62}$$

To get a relation like eq. (61), the nonlinear terms in eq. (62) have to be cancelled. This takes place if

$$4c_3 = c_2 \cdot c_4 \tag{63}$$

and

$$2c_3 = -c_4 \left(\frac{c_2}{2} + c_4 \right) \,. \tag{64}$$

$c_2 = 0$ implies $c_i = 0$ for all i, which is a non-interesting case. Therefore, we assume $c_2 \neq 0$. We are free to redefine the vector A_{β} by multiplying it with any non-vanishing constant, because we can compensate this by an

appropriate redefinition of the constants c_i, see eq. (50). We use this freedom to choose the value c_2 from eq. (52), i.e., $c_2 = 2$. Then eqs. (63), (64) read:

$$4c_3 = 2c_4 \, ; \qquad 2c_3 = -c_4(1 + c_4) \, .$$

The solution with $c_4 = 0$, which implies also $c_1 = c_3 = 0$, turns eq. (60) into

$$\Box A_\beta = -A^\gamma_{;\beta\gamma} \, .$$

This special case shall now be omitted, then we get as the only other solution $c_4 = -2$, $c_3 = -1$, and with eq. (57) finally $c_1 = -2$. This is together just the original quadruple eq. (52). So, we have shown, that eq. (61) indeed follows without further assumptions from the divergence of eq. (8), and we have shown in which sense the coefficients c_i are uniquely determined.

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